2017-2018 MM2MS2 Exam Solutions

1.

(a)

$$\delta_{total} = \delta_{thermal} + \delta_{mech} = 0$$

Therefore,

$$0 = L\alpha\Delta T + \frac{FL}{AE}$$

Rearranging,

$$F = -\alpha\Delta TAE = -22 \times 10^{-6} \times 25 \times (20 \times 10^{-3} \times 30 \times 10^{-3}) \times 70 \times 10^{9} = \mathbf{-23100\ N}$$

Therefore,

$$\sigma = \frac{F}{A} = \frac{-24150}{(20 \times 10^{-3} \times 30 \times 10^{-3})} = -4.025 \times 10^{7}\text{Pa} = \mathbf{-38.5\ MPa}$$

[5 marks]

(b)

As before:

$$\delta_{total} = \delta_{thermal} + \delta_{mech} = 0$$

where:

$$\delta_{mech} = \frac{FL}{AE}$$

However, for a small slice of the bar having a length dx, if allowed to expand freely:

$$d(\delta_T) = (\alpha\Delta T)dx$$

where:

$$\Delta T = \frac{40x}{L}$$

This gives:

$$\delta_T = \int_0^L \alpha \left(\frac{40x}{L}\right) dx = \alpha \int_0^L \left(\frac{40x}{L}\right) dx$$

or:

$$\delta_T = \alpha \left(\frac{40x^2}{2L}\right)\Bigg|_0^L = \alpha \frac{40L}{2}$$

Therefore:

$$0 = \alpha \frac{40L}{2} + \frac{FL}{AE}$$

and,

$$F = -\alpha \frac{40AE}{2}$$

which gives:

$$F = -\alpha \frac{40AE}{2} = -22 \times 10^{-6} \times \frac{40 \times (20 \times 10^{-3} \times 30 \times 10^{-3}) \times 70 \times 10^{9}}{2} = \mathbf{-18480\ N}$$

Therefore, the stress in the bar can be determined as:

$$\sigma = \frac{F}{A} = \frac{-19320}{(20 \times 10^{-3} \times 30 \times 10^{-3})} = -3.22 \times 10^{7}\text{Pa} = \mathbf{-30.8\ MPa}$$

[8 marks]

(c)

The ΔT through the beam is given by:

$$\Delta T = \frac{40y}{d}$$

As the bar is constrained, $\bar{\varepsilon} = 0$ and $1\backslash R = 0$

Considering the axial force equilibrium:

$$P = E\bar{\varepsilon}A - E\alpha\int_A \left(\frac{40y}{d}\right)dA$$

As the mean strain $\bar{\varepsilon} = 0$

$$P = 0 - E\alpha\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(\frac{40y}{d}\right)bdy = -\frac{40E\alpha b}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} ydy$$

$$\therefore P = -\frac{40E\alpha b}{d}\left[\frac{y^2}{2}\right]_{-\frac{d}{2}}^{\frac{d}{2}} = 0$$

Considering the moment equilibrium:

$$M = \frac{EI}{R} - E\alpha\int_A \left(\frac{40y}{d}\right)ydA$$

As $1\backslash R = 0$

$$M = 0 - E\alpha\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(\frac{40y^2}{d}\right)bdy = -\frac{40E\alpha b}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} y^2dy = -\frac{40E\alpha b}{d}\left[\frac{y^3}{3}\right]_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= -\frac{40E\alpha b}{d}\left[\frac{d^3}{24} - \frac{-d^3}{24}\right] = -\frac{10E\alpha bd^2}{3} = -\frac{10 \times 70 \times 10^9 \times 22 \times 10^{-6} 20 \times 10^{-3} \times (30 \times 10^{-3})^2}{3}$$

$$\therefore \boldsymbol{M = 92.4\ \mathrm{Nm}}$$

Using:

$$\sigma_x = E\left(\bar{\varepsilon} + \frac{y}{R} - \alpha\Delta R\right)$$

and considering $\bar{\varepsilon} = 0$ and $1\backslash R = 0$ gives:

$$\sigma_x = -E\alpha\Delta T$$

Substituting in for the temperature variation gives:

$$\sigma_x = -E\alpha\frac{40y}{d}$$

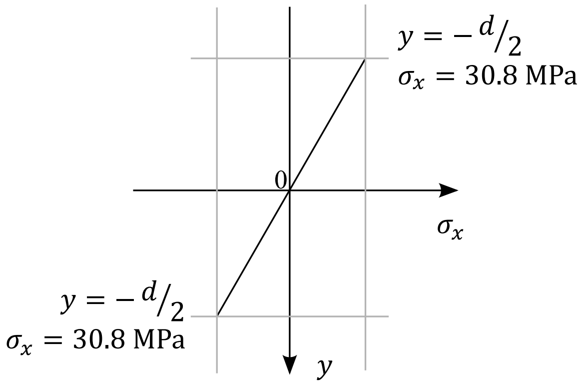
Evaluating at $y = d/2$ gives:

$$\sigma_x = -E\alpha\frac{40d}{2d} = -20E\alpha = -20 \times 70 \times 10^9 \times 22 \times 10^{-6} = -3.22 \times 10^7\,\text{Pa} = \mathbf{-30.8\ MPa}$$

and at $y = -d/2$:

$$\sigma_x = -E\alpha\frac{-40d}{2d} = 20E\alpha = 20 \times 70 \times 10^9 \times 22 \times 10^{-6} = 3.22 \times 10^7\,\text{Pa} = \mathbf{30.8\ MPa}$$

[10 marks]



[2 marks]

2.

(a)

Angle defined from the x-axis, ccw

Element 1 – Angle = 90 deg

[1 mark]

$$K_1^{el} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2e8 & 0 & -2e8 \\ 0 & 0 & 0 & 0 \\ 0 & -2e8 & 0 & 2e8 \end{bmatrix}$$

[3 marks]

Element 2 – Angle = 315 deg

[1 mark]

$$K_1^{el} = 1e7 \begin{bmatrix} 7.0711 & 7.07110 & -7.0711 & -7.0711 \\ 7.0711 & 7.0711 & -7.0711 & -7.0711 \\ -7.0711 & -7.0711 & 7.0711 & 7.0711 \\ -7.0711 & -7.0711 & 7.0711 & 7.0711 \end{bmatrix}$$

[3 marks]

Overall stiffness matrix

$$\boldsymbol{K} = \mathbf{1e8} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0.7071} & \mathbf{0.7071} & \mathbf{-0.7071} & \mathbf{-0.7071} \\ \mathbf{0} & \mathbf{-2} & \mathbf{0.7071} & \mathbf{2.7071} & \mathbf{-0.7071} & \mathbf{-0.7071} \\ \mathbf{0} & \mathbf{0} & \mathbf{-0.7071} & \mathbf{-0.7071} & \mathbf{0.7071} & \mathbf{0.7071} \\ \mathbf{0} & \mathbf{0} & \mathbf{-0.7071} & \mathbf{-0.7071} & \mathbf{0.7071} & \mathbf{0.7071} \end{bmatrix}$$

[4 marks]

If the applied load, *F*, is 30 kN:

(b)

Apply BCs

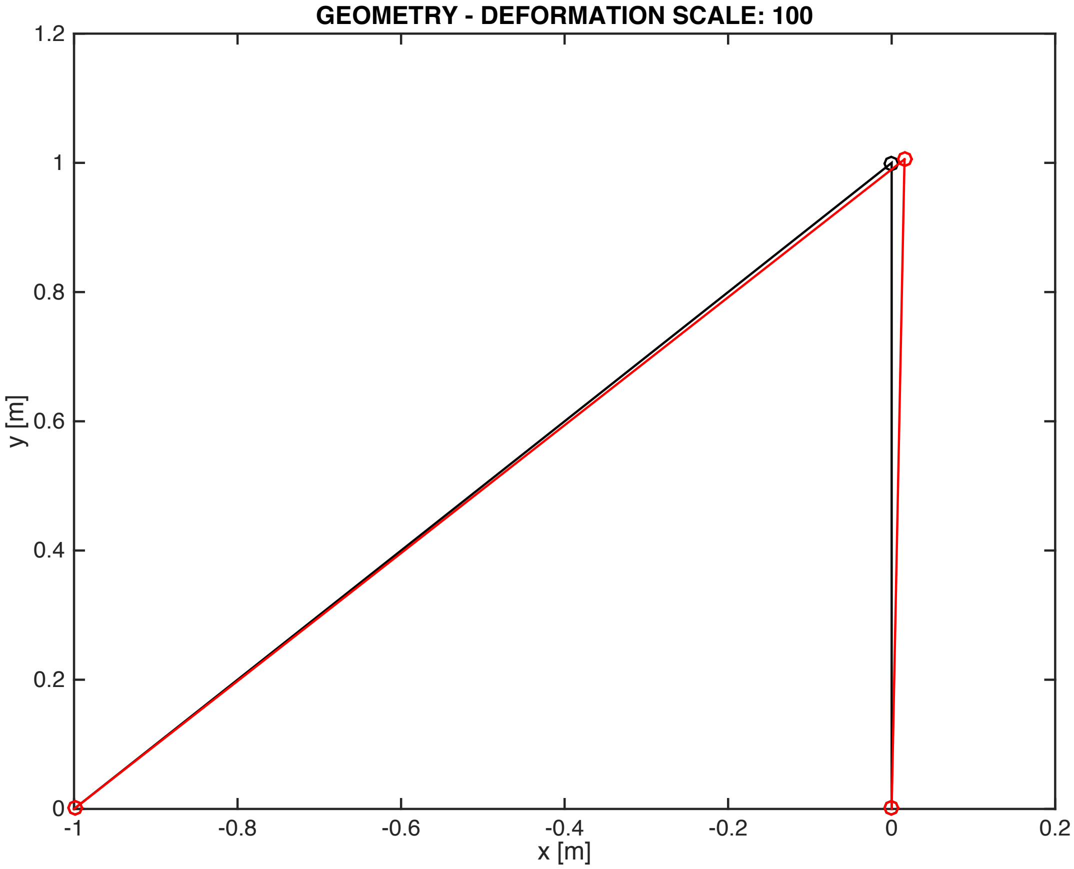
[2 marks]

Resolve Force

[2 marks]

$$U = 1\mathrm{e}-3\begin{bmatrix} 0 \\ 0 \\ \mathbf{0.1572} \\ \mathbf{0.0549} \\ 0 \\ 0 \end{bmatrix}$$

[3 marks]



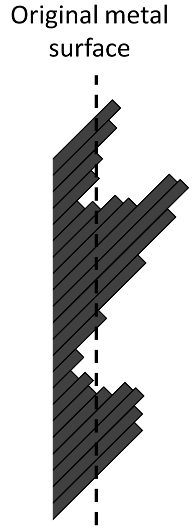
$$F = 1\mathrm{e}4\begin{bmatrix} 0 \\ \mathbf{-1.09810} \\ 1.5 \\ 2.5981 \\ -1.5 \\ \mathbf{-1.5} \end{bmatrix}$$

[6 marks]

3.

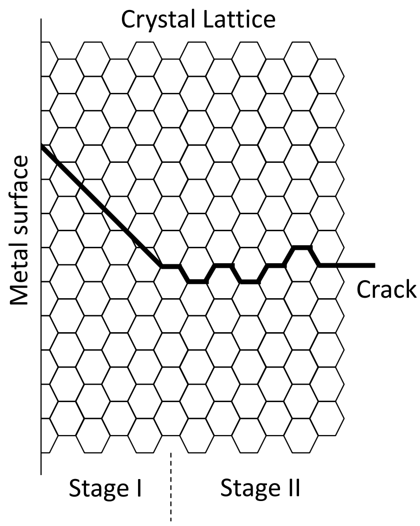
(a)

Crack Initiation and Stage I crack Growth: Pile up of dislocations causes slip bands, creating stress concentrations (features). This leads to shear stress controlled transgranular cracking. This occurs on the plane on maximum shear (45° to the loading plane) as shown below.



[3 marks]

Stage II Crack Growth: Once the crack has reached a critical length, the stress state around the crack tip changes and cracks will propagate due to the maximum tensile stress (90° to the loading direction). This phase is usually intergranular.

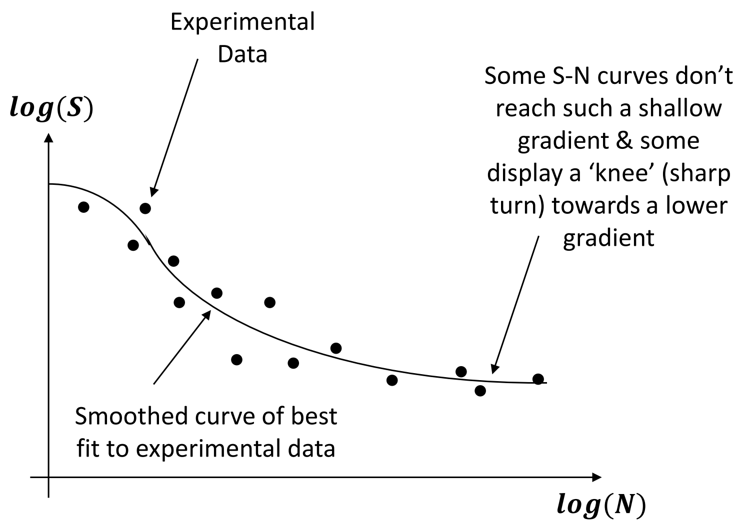


[4 marks]

Failure: Failure is at a critical crack length when the structure can no longer support the applied loads and it fails due to ductile tearing or cleavage (brittle) fracture.

[3 marks]

(b)

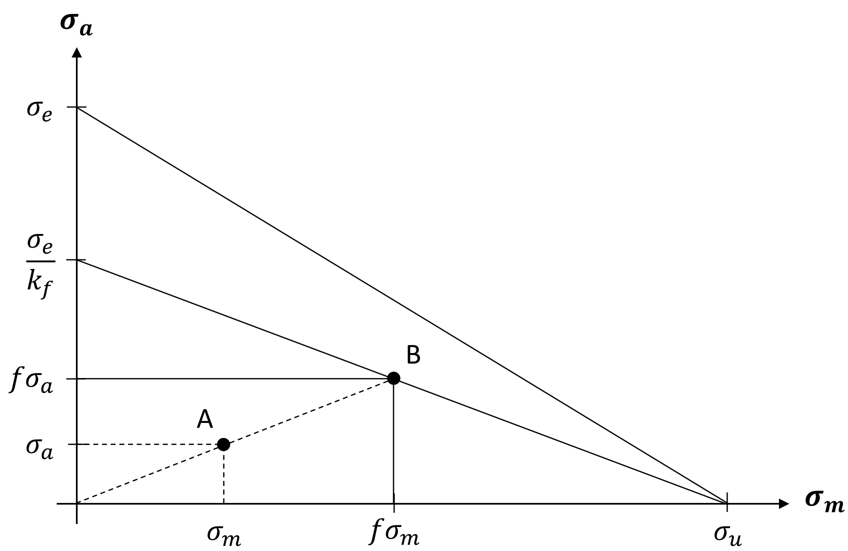


Fatigue strength is a hypothetical value of stress range at failure for exactly N cycles as obtained from an S-N curve.

The fatigue limit (sometimes called the endurance limit) is the limiting value of the median fatigue strength as N becomes very large, e.g. >10^8 cycles.

[5 marks]

(c)



[4 marks]

From similar triangles:

$$\frac{S_e}{k_f S_u} = \frac{f S_a}{S_u - f S_m}$$

[3 marks]

$$\therefore S_a = \frac{S_e(S_u - fS_m)}{fk_f S_u} = \frac{135(520 - 1.5 \times 80)}{1.5 \times 1.85 \times 520}$$

$$\therefore \boldsymbol{S_a = 37.42\ \mathrm{MPa}}$$

[3 marks]

4.

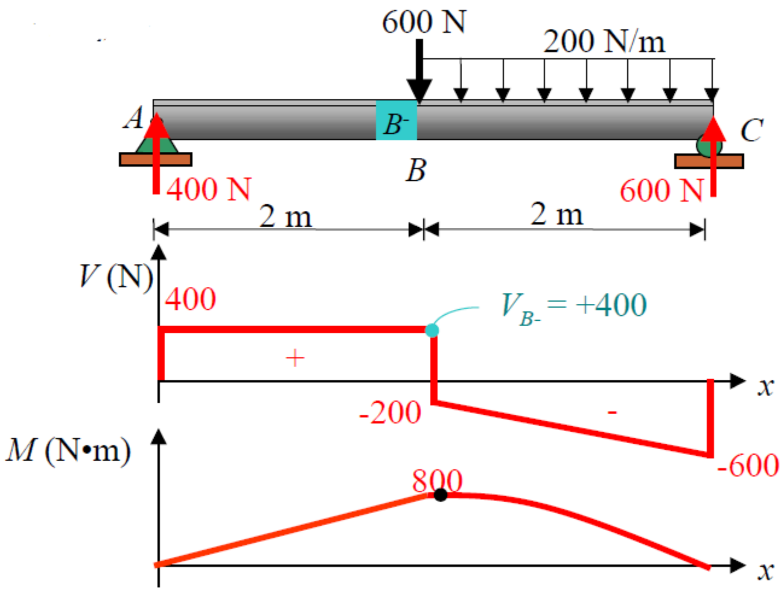
(a)

The reaction forces at A and C,

$$R_a + R_c = 600 + 200 \times 2$$

$$R_C \times 4 = 600 \times 2 + 400 \times 3$$

Therefore, the reaction forces $R_a = 400$ N, $R_c = 600$ N



The **maximum bending moment is 800 Nm located at the mid-length of the beam**.

[5 marks]

(b)

The second moment of area and polar second moment of area

$$I = \frac{\pi D^4}{64} = 3.14 * \frac{0.2^4}{64} = 7.85 * 10^{-5} m^4$$

$$J = 2I = 1.57 * 10^{-4} m^4$$

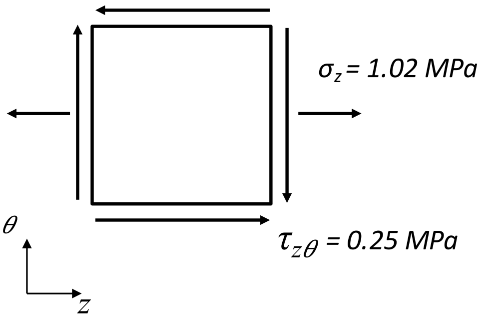
The tensile bending stress (axial stress) is

$$\sigma_z = \frac{My}{I} = 800 * \frac{0.1}{7.85 * 10^{-5}} = \mathbf{1.02\ MPa}$$

The shear stress is

$$\tau_{z\theta} = \frac{TR}{J} = 400 * \frac{0.1}{1.57 * 10^{-4}} = \mathbf{0.25\ MPa}$$

[7 marks]



[3 marks]

(c)

$$C = \frac{\sigma_z}{2} = \frac{1.02}{2} = 0.51 \text{ MPa}$$

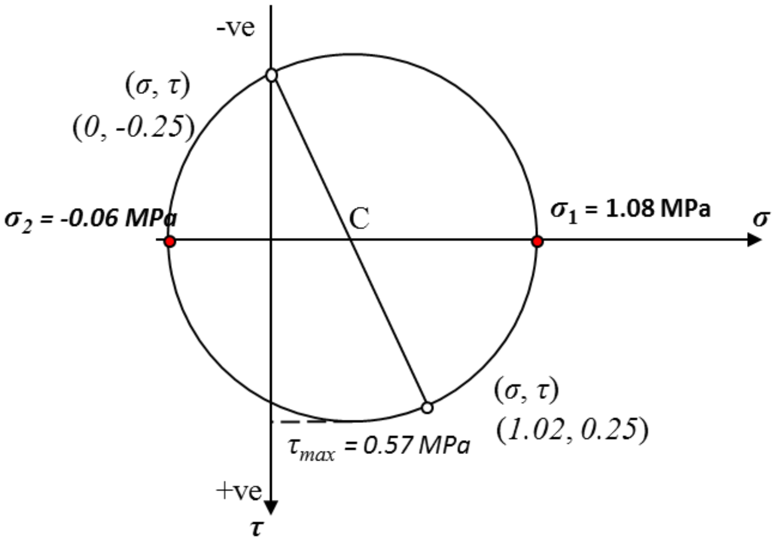
$$R = \sqrt{\left(\frac{\sigma_z}{2}\right)^2 + \tau_{z\theta}^2} = \sqrt{0.51^2 + 0.25^2} = 0.57 \text{ MPa}$$

$$\sigma_1 = C + R = 0.51 + 0.57 = \mathbf{1.08\ MPa}$$

$$\sigma_2 = C - R = 0.51 - 0.57 = \mathbf{0.06\ MPa}$$

$$\tau\_\text{max} = R = \mathbf{0.57\ MPa}$$

[7 marks]


-ve
(σ, τ)
$(0, -0.25)$
$\sigma_2 = -0.06$ MPa
C
$\sigma_1 = 1.08$ MPa
σ
(σ, τ)
$(1.02, 0.25)$
$\tau_{max} = 0.57$ MPa
+ve
τ

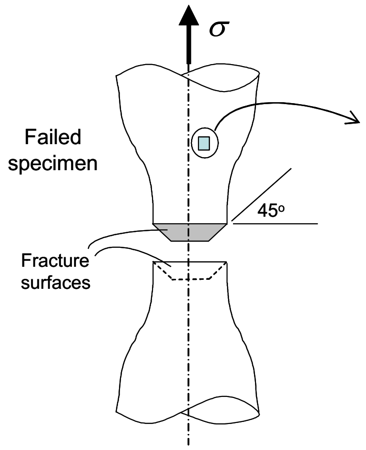
[3 marks

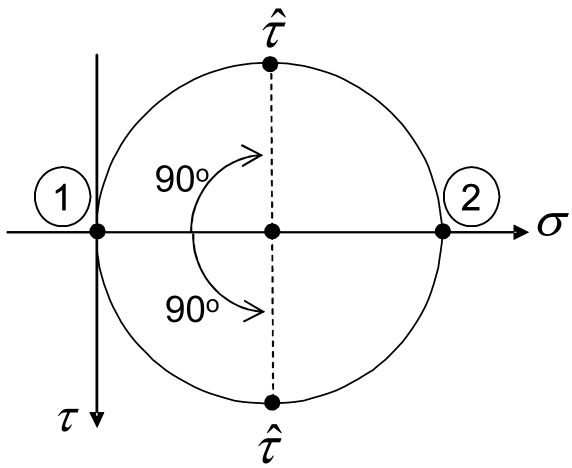
]

5.

(a)

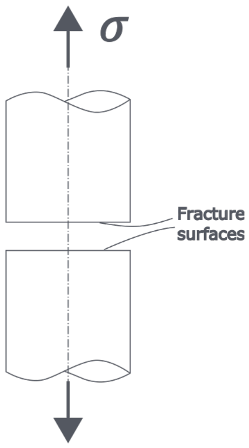
- If a tensile test is performed on a ductile material, the failure tends to be a cup and cone failure
- With a cone angle of 45°
- Failure occurs on the plane of maximum shear stress





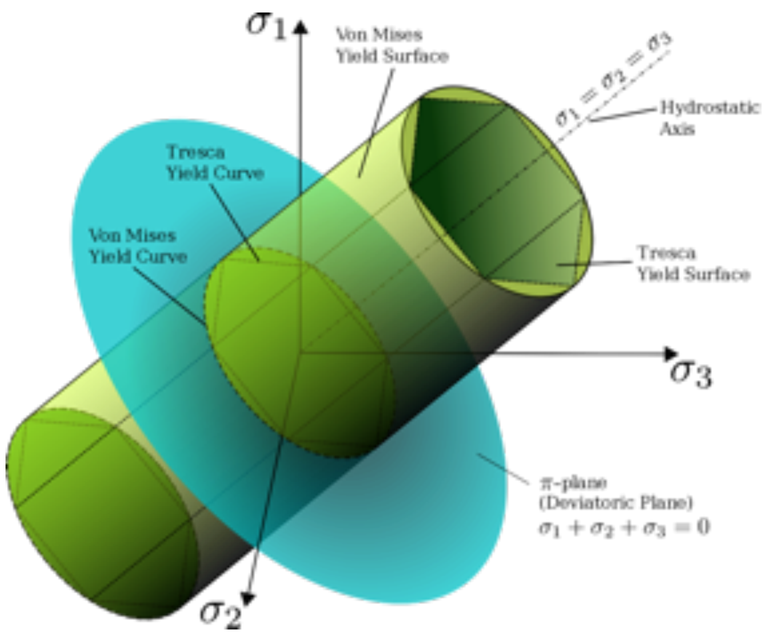
[3 marks]

- The failure of a tensile test of a brittle material, such as grey cast iron, will tend to be a flat fracture surface perpendicular to the loading direction
- Failure occurs on the plane of maximum principal stress

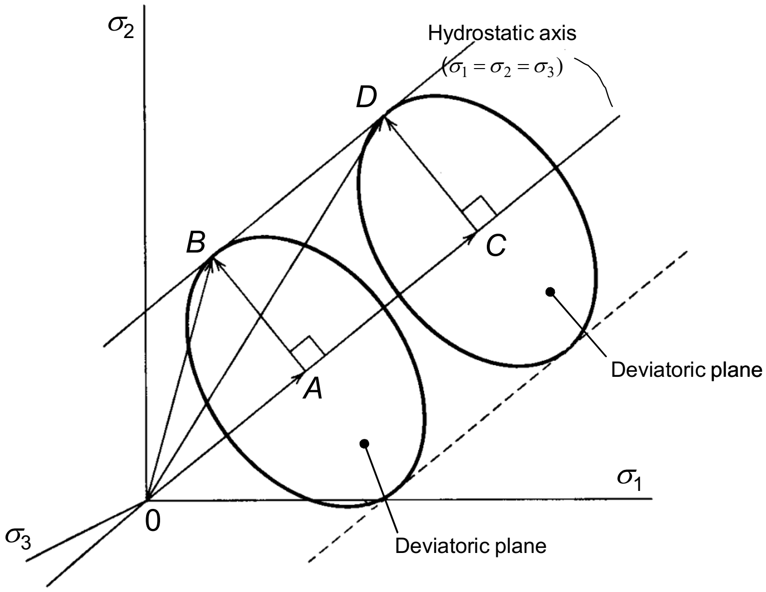


[3 marks]

(b)



[3 marks]



$$\sigma_h = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Deviatoric stress is given by:

$$\sigma' = (\sigma_1 - \sigma_h, \sigma_2 - \sigma_h, \sigma_3 - \sigma_h)$$

[3 marks for well explained, complete explanation, not just recalling the diagram, this only gets 2]

(c)

$$T = 8000 \text{ Nm}$$

$$M = 2000 \text{ Nm}$$

$$\sigma_y = 250 \text{ MPa}$$

Including safety factor (incorporated into yield stress):

$$\sigma_{y(SF)} = 125 \text{ MPa}$$

[1 mark]

$$\tau = \frac{Tr}{J}$$

$$\sigma_A = \frac{My}{I}$$

where $y = r$ in this case, therefore:

$$\sigma_A = \frac{Mr}{I}$$

$$J = \frac{\pi r^4}{2}$$

$$I = \frac{\pi r^4}{4}$$

Combining gives:

$$\sigma_A = \frac{4M}{\pi r^3}$$

$$\tau = \frac{2T}{\pi r^3}$$

[4 marks]

For Tresca:

$$\tau_{max} = \frac{\sigma_y}{2} = \frac{\sigma_{y(SF)}}{2} = \frac{125}{2} = 62.5 \text{ MPa}$$

therefore,

$$62.5 \times 10^6 = \tau_{max} = R = \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau^2}$$

$$R^2 = \left(\frac{2M}{\pi r^3}\right)^2 + \left(\frac{2T}{\pi r^3}\right)^2$$

$$(62.5 \times 10^6)^2 = \frac{4M^2}{\pi^2 r^6} + \frac{4T^2}{\pi^2 r^6}$$

$$(62.5 \times 10^6)^2 = \frac{2.72 \times 10^8}{\pi^2 r^6}$$

$$r^6 = \frac{2.72 \times 10^8}{\pi^2 (62.5 \times 10^6)^2}$$

$$r^6 = 7.05 \times 10^{-9}$$

$$r = 0.0438 \text{ m} = 43.8 \text{ mm}$$

$$\boldsymbol{d_{Tresca} = 87.6 \text{ mm}}$$

[4 marks]

For von Mises:

2D plane stress

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$$

C is centre of Mohr's circle, R is radius, expanding and tidying gives:

$$C^2 + 3R^2 = \sigma_y^2$$

$$\left(\frac{2M}{\pi r^3}\right)^2 + 3 \times \left(\left(\frac{2M}{\pi r^3}\right)^2 + \left(\frac{2T}{\pi r^3}\right)^2\right) = (125 \times 10^6)^2$$

$$\left(\frac{4M^2}{\pi^2 r^6}\right) + \left(\frac{12M^2}{\pi^2 r^6}\right) + \left(\frac{12T^2}{\pi^2 r^6}\right) = (125 \times 10^6)^2$$

$$\frac{16M^2 + 12T^2}{\pi^2 r^6} = (125 \times 10^6)^2$$

$$r^6 = \frac{16M^2 + 12T^2}{\pi^2 (125 \times 10^6)^2}$$

$$r^6 = 5.40 \times 10^{-9}$$

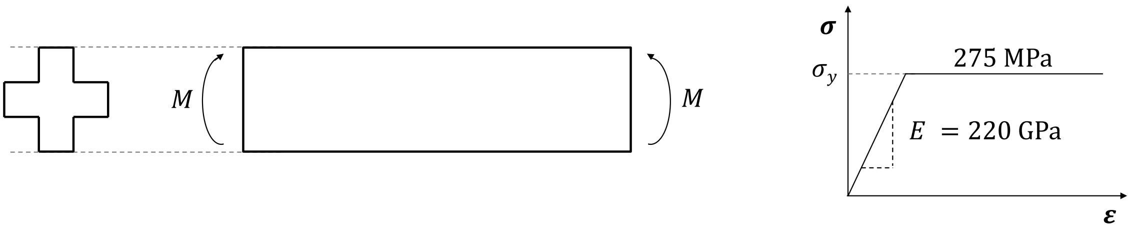
$$r = 0.0419 \text{ m} = 41.9 \text{ mm}$$

$$\boldsymbol{d_{von\ Mises} = 83.8 \text{ mm}}$$

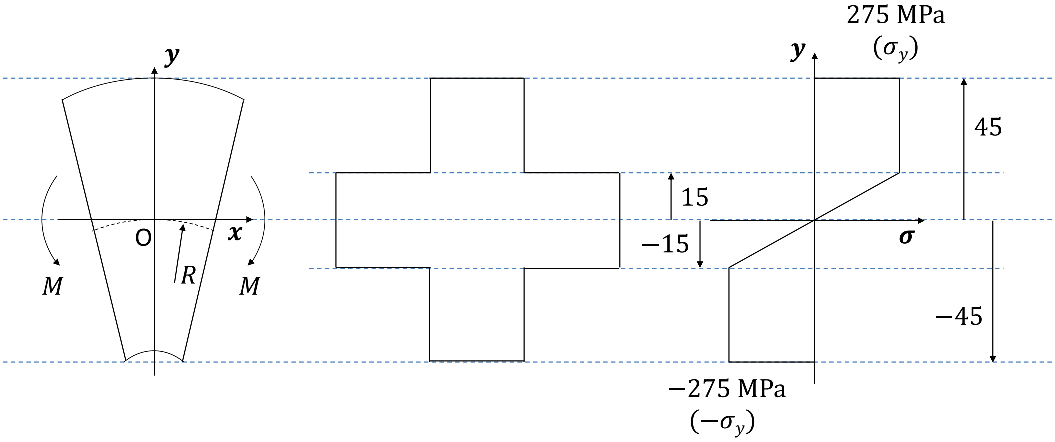
[4 marks]

6.

(a)



Yielding will occur through whole of the upper and lower $30\ \text{mm}$ (from top edge and bottom edge, respectively), therefore:



Variation of stress with y:

- For $45 < y < 15$, $\sigma = 275\ \text{MPa}$
- For $-15 < y < 15$, $\sigma = \frac{275}{15} y\ \text{MPa}$
- For $-45 < y < -15$, $\sigma = -275\ \text{MPa}$

[2 marks]

Moment equilibrium:a

(Balance the moments due to stresses in the elastic and plastic regions with the applied moment):

$$M = \int_A y\sigma dA = \int_y y\sigma b dy$$

[2 marks]

Due to the symmetry of the stress distribution and substituting in the elastic and plastic terms for σ, this can be rewritten as:

$$M = 2\left\{\int_0^{15} y\frac{275}{15}y(90)dy + \int_{15}^{45} y(275)(30)dy\right\}$$

$$= 2\left\{1{,}650\int_0^{15} y^2dy + 8{,}250\int_{15}^{45} ydy\right\}$$

$$= 2\left\{1{,}650\left[\frac{y^3}{3}\right]_0^{15} + 8{,}250\left[\frac{y^2}{2}\right]_{15}^{45}\right\}$$

$$= 2\left\{1{,}650\left(\frac{15^3}{3}\right) + 8{,}250\left(\frac{45^2}{2} - \frac{15^2}{2}\right)\right\}$$

$$= 2\{1{,}856{,}250 + 7{,}425{,}000\}$$

$$\therefore \boldsymbol{M = 18{,}562{,}500\ \text{Nmm} = 18.56\ \text{kNm}}$$

[3 marks]

Compatibility:

$$\varepsilon = \frac{y}{R} \qquad (1)$$

[2 marks]

At $y = 15$ mm, $\sigma = \sigma_y = 275$ MPa and since this point is within the elastic range:

$$\varepsilon = \frac{\sigma_y}{E} = \frac{275}{220{,}000} = 1.25 \times 10^{-3}$$

[2 marks]

Substituting this into (1) gives:

$$1.25 \times 10^{-3} = \frac{15}{R}$$

$$\therefore \boldsymbol{R = 12{,}000\ \text{mm} = 12\ \text{m}}$$

[2 marks]

(b)

$$I_{mid} = \left(\frac{bd^3}{12}\right)_{mid} = \frac{90 \times 30^3}{12} = 202{,}500 \text{ mm}^2$$

[2 marks]

Using the parallel axis theorem:

$$I_{top}{}' = I_{top} + Ah^2 = \left(\frac{30 \times 30^3}{12}\right) + (30 \times 30) \times 30^2 = 67{,}500 + 810{,}000 = 877{,}500 \text{ mm}^4$$

[1 mark]

and since $I_{bottom}{}' = I_{top}{}'$:

$$I = I_{mid} + 2I_{top}{}' = 1{,}957{,}500 \text{ mm}^2$$

[1 mark]

Unloading is assumed to be entirely elastic. Beam bending equation:

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta \sigma}{y}$$

[1 mark]

Max change in stress ($\Delta\sigma$) will occur at $y\ = \frac{d}{2} = y_{max}\ (= \pm 45 \text{ mm})$.

$$\therefore \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-18{,}562{,}500 \times \pm 45}{1{,}957{,}500}$$

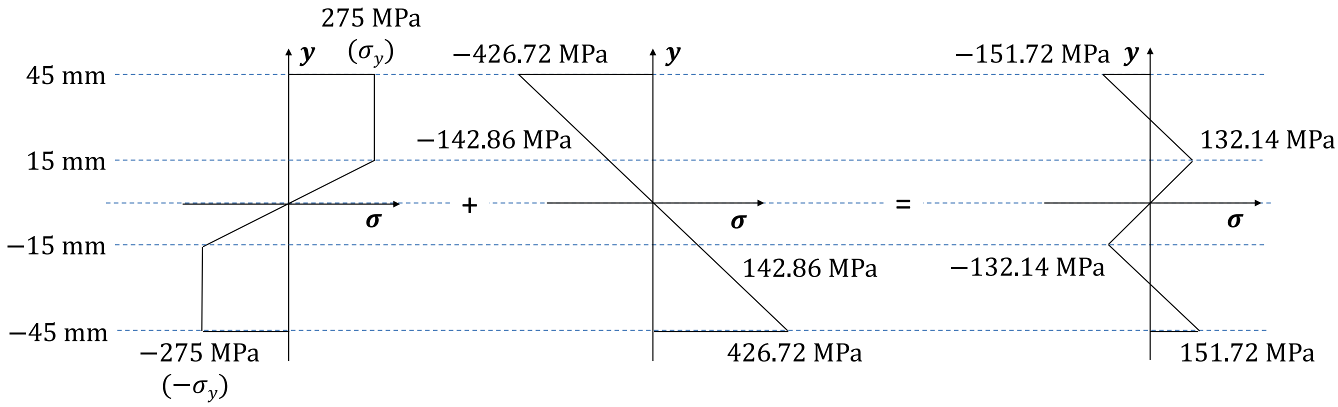
$$= \mp 426.72 \text{ MPa}$$

i.e.:

$$at\ y = 45 \text{ mm}, \therefore \Delta\sigma_{max}^{el} = -426.72 \text{ MPa}$$

$$and\ at\ y = -45 \text{ mm}, \therefore \Delta\sigma_{max}^{el} = 426.72 \text{ MPa}$$

[1 mark]



Loading stress distribution

(Elastic) unloading stress distribution

Residual stress distribution

Interpolation of (elastic) unloading line:

At $y = 45$ mm, $\sigma = -426.72$ MPa

$$y = m\sigma + c$$

$$\therefore 45 = \text{m} \times -426.72 + 0$$

$$\therefore \text{m} = -0.105$$

At $y = 15$ mm, $15 = -0.105 \times \sigma$

$$\therefore \sigma = -142.86 \text{ MPa}$$

[3 marks]

Residual stress is below yield (275 MPa), so reverse yielding does not occur. At $y = 15$ mm, no plastic deformation occurs during loading and unloading,

$$\therefore \varepsilon_{residual} = \frac{\sigma_{residual}}{E} = \frac{132.14}{220{,}000} = 6.006 \times 10^{-4}$$

[1 mark]

Also,

$$\varepsilon = \frac{y}{R}$$

$$\therefore 6.006 \times 10^{-4} = \frac{15}{R}$$

$$\therefore \boldsymbol{R = 24{,}975.02 \text{ mm} = 24.98 \text{ m}}$$

[2 marks]